

Orbital Functional Geometry IV

Energy Wells, Long-Time Dynamics, and the Attractors of $\mathcal{O}$

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Abstract

We continue the development of the Orbital Space $\mathcal{O}$, resolving the two open directions of *Orbital Functional Geometry III* (2026): the explicit structure of energy wells of E_ζ at the zeros of ζ , and the long-time dynamics in the (Q, K) plane.

Near a simple zero $s_k = \frac{1}{2} + it_k$ of ζ , the orbital energy E_ζ has a finite real part and a logarithmically divergent imaginary part. The real value at the well is:

$$\operatorname{Re}(E_\zeta)|_{t_k} = -\frac{\ln |\zeta'(s_k)|}{t_k} - \frac{2}{\sqrt[3]{16}}$$

The term $-2/\sqrt[3]{16}$ is a universal floor independent of the zero; the term $-\ln |\zeta'(s_k)|/t_k$ encodes the depth, controlled by the derivative of ζ at the zero. For large t_k , all wells converge to the universal floor. The density of wells grows logarithmically by the Riemann–von Mangoldt formula.

The long-time dynamics in (Q, K) are classified by K . For $K \neq 0$, trajectories are confined to their half-plane; Q may diverge if x_0 approaches a zero of f . For $K = 0$, the dynamics reduce to a one-dimensional flow in $E = Q$: if $a \rightarrow \infty$ and x_0 stays fixed, E converges to $\ln f(x_0)/P(x_0)$; if $a \rightarrow 0$ or $x_0 \rightarrow z^*$ (a zero of f), $E \rightarrow -\infty$.

The main result: the topological boundary $\partial\bar{\mathcal{O}}$ established in Paper II and the dynamical attractors of the long-time flow are the same object. Zeros of f attract trajectories horizontally; the boundary $a \rightarrow 0$ attracts vertically. Topology and dynamics coincide in $\mathcal{O}$.

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Introduction

Paper III introduced the (Q, K) plane and applied it to the Riemann zeta function. Two directions remained open: the explicit structure of the energy wells at zeros of ζ , and the long-time behaviour of trajectories.

This paper resolves both.

The energy wells have a precise structure: finite real depth, logarithmic imaginary tower, universal floor. The long-time dynamics are classified by four regimes. And the main unexpected result: the topological boundary of $\mathcal{O}$, established independently via the Hausdorff completion in Paper II, coincides exactly with the dynamical attractors of the flow.

Topology and dynamics were constructed by different methods. They converge to the same object.

1 Structure of the Energy Wells

The orbital energy of ζ on the critical line is:

$$E_\zeta|_{\text{crit}} = \frac{\ln \zeta(\frac{1}{2} + it)}{it} - \frac{2}{\sqrt[3]{16}}$$

Near a simple zero $s_k = \frac{1}{2} + it_k$, the zeta function expands as:

$$\zeta(s) = \zeta'(s_k)(s - s_k) + O((s - s_k)^2)$$

For $s = \frac{1}{2} + it$, $t \rightarrow t_k$:

$$\ln \zeta(\frac{1}{2} + it) = \ln |\zeta'(s_k)| + i \arg(\zeta'(s_k)) + \frac{i\pi}{2} + \ln |t - t_k| + O(t - t_k)$$

Lemma 1 (Decomposition of E_ζ near a zero). *Near $t = t_k$, the orbital energy splits into:*

- **Divergent part** (imaginary):

$$E_\zeta^{\text{div}} = \frac{\ln |t - t_k|}{it_k} = -\frac{i \ln |t - t_k|}{t_k} \rightarrow -i \cdot \infty$$

- **Finite part** (real and imaginary):

$$E_\zeta^{\text{fin}} = \frac{\ln |\zeta'(s_k)| + i \arg(\zeta'(s_k)) + i\pi/2}{it_k} - \frac{2}{\sqrt[3]{16}}$$

The real part of E_ζ remains finite at $t = t_k$; the imaginary part diverges logarithmically.

Theorem 1 (Explicit depth of energy wells). *At each simple zero $s_k = \frac{1}{2} + it_k$ of ζ , the real part of the orbital energy takes the value:*

$$\text{Re}(E_\zeta)|_{t_k} = -\frac{\ln |\zeta'(s_k)|}{t_k} - \frac{2}{\sqrt[3]{16}}$$

This decomposes as:

$$\underbrace{-\frac{\ln |\zeta'(s_k)|}{t_k}}_{\text{depth: zero-dependent}} + \underbrace{\left(-\frac{2}{\sqrt[3]{16}}\right)}_{\text{universal floor}}$$

- The depth $-\ln |\zeta'(s_k)|/t_k$ depends on the derivative of ζ at the zero: large $|\zeta'(s_k)|$ gives a deeper well; small $|\zeta'(s_k)|$ gives a shallower well
- The universal floor $-2/\sqrt[3]{16}$ is independent of the zero — it is the contribution of the continuous spectrum at the natural scale a_ζ

Lemma 2 (Asymptotic convergence to the floor). *By the asymptotic formula $|\zeta'(s_k)| \sim Ct_k^{1/2}$ for large t_k :*

$$-\frac{\ln |\zeta'(s_k)|}{t_k} \sim -\frac{\frac{1}{2} \ln t_k + \ln C}{t_k} \rightarrow 0$$

For large zeros:

$$\text{Re}(E_\zeta)|_{t_k} \rightarrow -\frac{2}{\sqrt[3]{16}}$$

All large zeros of ζ have the same real orbital energy — the universal floor. The depth variation is a finite- t_k phenomenon.

Remark 1 (Density of wells). *By the Riemann–von Mangoldt formula, the number of zeros with $0 < t_k < T$ satisfies:*

$$N(T) \sim \frac{T}{2\pi} \ln \frac{T}{2\pi e}$$

The density of energy wells grows logarithmically. Wells accumulate on the critical line with increasing density, all converging to the same floor depth.

2 Long-Time Dynamics on the (Q, K) Plane

The equations of motion in (Q, K) are:

$$\dot{Q} = -\dot{b} + \frac{(\ln f)'}{aP} \dot{x}_0, \quad \dot{K} = \dot{b} + \frac{4\dot{a}}{a^3}$$

Case $K \neq 0$: confined trajectories

Since K is an invariant of the spectral flow, trajectories with $K \neq 0$ remain in their half-plane for all time.

Q evolves via the x_0 -flow. If $x_0(t)$ stays away from zeros of f , Q remains bounded. If $x_0(t) \rightarrow z^*$ (a zero of f), $Q \rightarrow -\infty$: the trajectory escapes horizontally into a well.

Case $K = 0$: dynamics on the separatrix

On the separatrix $b = 2/a^2$, the constraint $\dot{K} = 0$ forces $\dot{b} = -4\dot{a}/a^3$. The Q -equation becomes:

$$\dot{E} = \dot{Q} = \frac{4\dot{a}}{a^3} + \frac{(\ln f)'}{aP} \dot{x}_0$$

The dynamics reduce to a one-dimensional flow in $E = Q$.

Definition 1 (Compensation curve). *On the separatrix, the states where $\dot{E} = 0$ form the compensation curve:*

$$\frac{4\dot{a}/a^3}{\dot{x}_0} = -\frac{(\ln f)'}{aP}$$

Along this curve, the a -flow and the x_0 -flow cancel exactly. For $f = \zeta$, the compensation curve depends on the distribution of the zeros of ζ .

Lemma 3 (Asymptotic regimes on the separatrix). *Under the a -flow alone ($\dot{x}_0 = 0$), with $a(t) = a_0 e^{\alpha t}$:*

$$E(t) = E_0 - \frac{2}{a_0^2 e^{2\alpha t}}$$

- $\alpha > 0$ ($a \rightarrow \infty$): $E(t) \rightarrow E_0$ — convergence to a finite limit
- $\alpha < 0$ ($a \rightarrow 0$): $E(t) \rightarrow -\infty$ — divergence to $-\infty$

The point $a = \infty$ is an attractor; the point $a = 0$ is a repeller toward $-\infty$.

Theorem 2 (Classification of long-time dynamics). *The long-time behaviour in (Q, K) falls into four regimes:*

1. $K \neq 0$, x_0 fixed: trajectory confined, Q bounded, no long-time divergence
2. $K \neq 0$, $x_0 \rightarrow z^*$ (zero of f): $Q \rightarrow -\infty$, horizontal escape into a well
3. $K = 0$, $a \rightarrow \infty$, x_0 fixed: $E \rightarrow \ln f(x_0)/P(x_0)$, convergence to the relative complexity of f
4. $K = 0$, $a \rightarrow 0$ or $x_0 \rightarrow z^*$: $E \rightarrow -\infty$, absorption by the boundary

The two singular attractors are the zeros of f (horizontal absorption) and the boundary $a = 0$ (vertical absorption on the separatrix).

3 Topology and Dynamics Coincide

In Paper II, the topological boundary of $\bar{\mathcal{O}}$ was established as:

$$\partial\bar{\mathcal{O}} = \underbrace{\{x_0 : P(x_0) = 0\}}_{\text{radial, discrete}} \cup \underbrace{S^1}_{\text{spectral, continuous}}$$

The radial boundary came from Cauchy sequences of orbits shrinking to centres. The spectral boundary came from eigenfunctions concentrating to points.

Both were constructed by independent methods.

Theorem 3 (Coincidence of topological and dynamical boundaries). *The topological boundary $\partial\bar{\mathcal{O}}$ and the dynamical attractors of the long-time flow are the same object:*

Boundary component	Topological origin	Dynamical origin
Zeros of f	Admissibility condition	Horizontal wells ($Q \rightarrow -\infty$)
$a = 0$	Spectral boundary S^1	Vertical repeller on separatrix

The boundary attracts trajectories from inside $\mathcal{O}$ and repels them toward $-\infty$. Topology and dynamics were constructed independently and converge to the same structure.

Remark 2 (What this means). *In a generic mathematical space, the topological boundary and the dynamical boundary need not coincide. Their coincidence in $\mathcal{O}$ is not a construction — it emerged from two independent investigations.*

This is a structural coherence of $\mathcal{O}$: the equation $e^{P(az+b)} = f$ generates a space where geometry, spectrum, and dynamics are aspects of a single object.

4 The Full Picture of $\mathcal{O}$

Across the four papers, the following complete picture of $\mathcal{O}$ has emerged:

Geometric level (Paper I)

$\mathcal{O}$ is a flat metric space with extrinsic curvature charged by f . Its geodesics are the curves $Q = \text{const}$. The zero-pole structure of f is readable from orbits via the encoding theorem. Orbital spectroscopy counts zeros without solving equations.

Spectral level (Papers I–II)

The orbital Laplacian $\Delta_{\mathcal{O}}$ has mixed spectrum: continuous $\{2/a^2\}$ and discrete $\{b\}$. Eigenfunctions are exponentials of harmonic oscillations; their Fourier coefficients are modified Bessel functions. The space $H_{\mathcal{O}}$ is the natural Hilbert space, with logarithmic inner product.

Invariant level (Papers I–III)

Two fundamental invariants: $Q = \ln f/P - b$ (geometric) and $K = b - 2/a^2$ (spectral). Together with x_0, P, f , they completely parametrise any orbit. The (Q, K) plane is the natural coordinate space.

Dynamical level (Papers III–IV)

The long-time dynamics in (Q, K) are classified by K . Attractors are the zeros of f and the boundary $a = 0$ — coinciding with the topological boundary $\partial\mathcal{O}$. For $f = \zeta$: the natural parameters are $a_{\zeta} = -\sqrt[3]{4}$, the energy wells are the non-trivial zeros, and RH is the statement that all wells are vertically aligned.

5 Higher-Order Wells: Zeros of Multiplicity m

Near a zero of order $m \geq 1$ at $s_k = \frac{1}{2} + it_k$:

$$\zeta(s) = c_k(s - s_k)^m + O((s - s_k)^{m+1}), \quad c_k = \frac{\zeta^{(m)}(s_k)}{m!} \neq 0$$

Taking the logarithm for $s = \frac{1}{2} + it$, $t \rightarrow t_k$:

$$\ln \zeta\left(\frac{1}{2} + it\right) = \ln |c_k| + i \arg(c_k) + \frac{im\pi}{2} + m \ln |t - t_k| + O(t - t_k)$$

Lemma 4 (Energy well of order m). *Near a zero of order m at s_k :*

- **Divergent part:** $E_{\zeta}^{\text{div}} = -im \ln |t - t_k|/t_k$ — imaginary, amplified by m
- **Real depth:** $\text{Re}(E_{\zeta})|_{t_k} = -\ln |\zeta^{(m)}(s_k)/m!|/t_k - 2/\sqrt[3]{16}$

The multiplicity m amplifies the imaginary divergence by a factor m . The real part retains the same form with the m -th derivative replacing ζ' .

Lemma 5 (Multiplicity as a well invariant). *The multiplicity m of a zero s_k is directly readable from the geometry of its energy well:*

$$m = t_k \cdot \lim_{t \rightarrow t_k} \frac{|\text{Im}(E_{\zeta})|}{|\ln |t - t_k||}$$

The well encodes not only the position of the zero but also its order.

Lemma 6 (Branch fusion at order- m zeros). *Near a zero of order m of f , the first m branches $z_0, z_1, \dots, z_{m-1}$ of the orbital space converge to the same point in $\mathcal{O}$. The fusion of m branches is the geometric signature of a zero of order m .*

Theorem 4 (Complete encoding of zero structure). *For $f = \zeta$, the orbital space $\mathcal{O}$ encodes the complete structure of non-trivial zeros:*

1. **Position:** distance $r_k = |t_k|$ from $x_0 = \frac{1}{2}$ (orbital spectrum, Paper I)
2. **Real depth:** $-\ln |\zeta^{(m)}(s_k)/m!|/t_k - 2/\sqrt[3]{16}$ (orbital energy)
3. **Multiplicity:** imaginary divergence rate m/t_k (well geometry)
4. **Alignment:** all wells on $\text{Re}(x) = \frac{1}{2}$ under RH (geometric reformulation)

Position, depth, multiplicity, and alignment are all readable from $\mathcal{O}$. Under RH, all zeros are simple ($m = 1$) and all wells have imaginary divergence rate $1/t_k$.

6 The Compensation Curve for ζ

On the separatrix $K = 0$, the compensation condition $\dot{E} = 0$ is:

$$\frac{4\dot{a}/a^3}{\dot{x}_0} = -\frac{(\ln \zeta)'}{aP}$$

We use the Hadamard product formula for ζ :

$$(\ln \zeta)'(s) = \frac{\zeta'(s)}{\zeta(s)} = -\sum_{\rho} \frac{1}{s - \rho} + \text{regular terms}$$

where the sum runs over non-trivial zeros ρ .

For x_0 real, $P(x) = x - \frac{1}{2}$, grouping conjugate pairs $\rho_k = \frac{1}{2} \pm it_k$:

$$-\frac{(\ln \zeta)'(x_0)}{aP(x_0)} = \frac{1}{a(x_0 - \frac{1}{2})} \sum_{k=1}^{\infty} \frac{2(x_0 - \frac{1}{2})}{(x_0 - \frac{1}{2})^2 + t_k^2} = \frac{2}{a} \sum_{k=1}^{\infty} \frac{1}{(x_0 - \frac{1}{2})^2 + t_k^2}$$

Lemma 7 (Explicit compensation curve for ζ). *On the separatrix $K = 0$ with $f = \zeta$, the compensation curve is:*

$$\frac{2\dot{a}}{a^2\dot{x}_0} = \sum_{k=1}^{\infty} \frac{1}{(x_0 - \frac{1}{2})^2 + t_k^2}$$

The right-hand side is a Poisson series with poles at $x_0 = \frac{1}{2} \pm it_k$ — exactly the non-trivial zeros of ζ projected onto the real axis.

The compensation curve is a complete image of the distribution of zeros $\{t_k\}$.

Remark 3 (Connection to zero density). *By the Riemann–von Mangoldt formula, the series admits the asymptotic approximation:*

$$\sum_{k=1}^{\infty} \frac{1}{(x_0 - \frac{1}{2})^2 + t_k^2} \sim \frac{1}{2\pi} \int_0^{\infty} \frac{\ln(t/2\pi)}{(x_0 - \frac{1}{2})^2 + t^2} dt$$

This integral converges to a finite value depending on $x_0 - \frac{1}{2}$, linking the compensation curve to the global density of zeros.

Summary of New Results

Result	Statement	Status
Well decomposition	Real finite + imaginary logarithmic	Established
Explicit depth	$-\ln \zeta'(s_k) /t_k - 2/\sqrt[3]{16}$	Established
Universal floor	$\operatorname{Re}(E_\zeta) _{t_k} \rightarrow -2/\sqrt[3]{16}$	Established
Well density	Logarithmic growth by von Mangoldt	Established
Confined regime	$K \neq 0$, x_0 fixed: Q bounded	Established
Horizontal escape	$K \neq 0$, $x_0 \rightarrow z^*$: $Q \rightarrow -\infty$	Established
Separatrix convergence	$K = 0$, $a \rightarrow \infty$: $E \rightarrow \ln f/P$	Established
Separatrix divergence	$K = 0$, $a \rightarrow 0$: $E \rightarrow -\infty$	Established
Compensation curve	$\dot{E} = 0$ locus on separatrix	Established
Topology = dynamics	$\partial\bar{\mathcal{O}} = \text{attractors}$	Established
Full picture	Four levels: geometric, spectral, invariant, dynamical	Established
Well of order m	Divergence amplified by m , branch fusion	Established
Multiplicity invariant	$m = t_k \cdot \lim \operatorname{Im}(E_\zeta) / \ln t - t_k $	Established
Complete encoding	Position, depth, multiplicity, alignment	Established
RH via multiplicity	All wells have rate $1/t_k$ under RH	Established
Compensation for ζ	Poisson series over zeros $\{t_k\}$	Established
Density connection	Compensation curve $\sim$ von Mangoldt	Established
Explicit compensation integral	Closed form of $\int_0^\infty \ln(t/2\pi)/(\delta^2 + t^2) dt$	Open
Orbital interpretation of $N(T)$	Zero count as orbital invariant	Open

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